MATH 732: CUBIC HYPERSURFACES

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1. INTRO TO FANO SCHEMES

These notes are based on [Huy23, §2.1]. See the disclaimer section.

Let $X \subseteq \mathbf{P}^N$ be a projective variety over k. The *Fano scheme of r-planes in X*, denoted $\mathrm{F}(X,r)$ represents the functor:

$$T \mapsto \left\{ L \subseteq T \times X \;\middle|\; \begin{array}{l} Z \text{ is flat over } T \text{ and for every point } t \in T \\ \text{the fiber } L_{k(t)} \subseteq \mathbf{P}^N_{k(t)} \text{ is an } r\text{-plane} \end{array} \right\} \Big/ \simeq .$$

In the case $X = \mathbf{P}^N$, then of course the functor $\mathrm{F}(\mathbf{P}^N, r)$ is represented by $\mathrm{Gr}(r+1, N+1)$, but this requires proof.

Proposition 1.1. $\mathrm{F}(\mathbf{P}^N, r) = \mathrm{Gr}(r+1, N+1)$.

Proof. On the one hand, $\mathbf{G} = \mathrm{Gr}(r+1, N+1)$ carries a tautological rank $r+1$ vector bundle:

$$\mathcal{S} \subseteq \mathcal{O}_{\mathbf{G}}^{\oplus N+1}.$$

This induces a linear inclusion of projective bundles (with the sub-space convention):

$$\Lambda := \mathbf{P}(\mathcal{S}) \subseteq \mathbf{P}(\mathcal{O}_{\mathbf{G}}^{\oplus N+1}) \simeq \mathbf{G} \times \mathbf{P}^N.$$

For any map $T \to \mathbf{G}$, there is an induced r-plane:

$$\Lambda \times_{\mathbf{G}} T \subseteq \mathbf{P} \times T.$$

This induces a map of functors:

$$\mathrm{Hom}(-, \mathbf{G}) \to \mathbf{F}(\mathbf{P}^N, r)(-).$$

Conversely (sketchy), given a relative r-plane:

$$L \subseteq T \times \mathbf{P}^N$$

we want to show there is an induced map $T \to \mathbf{G}$. One idea is try to construct a natural map to the Plücker embedding space:

$$T \to \mathbf{P}(\wedge^{r+1}(k^{\oplus N+1})).$$

If $p_1 \colon L \to T$ and $p_2 \colon L \to \mathbf{P}^N$ are the natural projections, then this can be proved by showing that the dual of the tautological subbundle:

$$\mathcal{O}_L^{\oplus N+1} \to p_1^*(\mathcal{O}_{\mathbf{P}^N}(1))$$

pushes forward to a rank $r+1$ quotient bundle:

$$\mathcal{O}_T^{\oplus N+1} \to (p_2)_*(p_1^*(\mathcal{O}(1))).$$

Taking $(r+1)$-th exterior powers induces the desired map to the Plücker space and it needs to be checked that the equations of the Grassmannian are satisfied (especially in the nonreduced case). We won't comment further. $\qquad\square$

Now in the case of a degree d hypersurface $X = (F = 0) \subseteq \mathbf{P} = \mathbf{P}^{n+1}$, the Fano scheme admits a concrete description in terms of the tautological bundle of $\mathbf{G}$. As we already said, the tautological bundle

$$\mathcal{S} \subseteq \mathcal{O}_{\mathbf{G}}^{\oplus n+2}$$

is the dual of the vector bundle:

$$(p_2)_*(p_1^*\mathcal{O}_{\mathbf{P}}(1)) = \mathcal{S}^\vee$$

and so degree 1 polynomials give rise to sections by pulling back and pushing forward the sections. Likewise, our degree d polynomial F gives rise to a section $s_F \in \mathrm{H}^0(\mathbf{G}, \mathrm{Sym}^d(\mathcal{S}^\vee))$.

Exercise 1. *Prove that a subscheme $T \subseteq \mathbf{G}$ is contained in $\mathrm{F}(X,r)$ if and only if $s_F|_T \equiv 0$. In other words:*

$$\mathrm{F}(X,r) = (s_F = 0) \subseteq \mathbf{G}.$$

Moreover, use the fact that $\mathrm{Sym}^d(\mathcal{S}^\vee)$ is globally generated to show that for general F the section s_F is transverse to the zero section (so it is smooth and its class computes the top Chern class of $\mathrm{Sym}^d \mathcal{S}^\vee$).

Remark 1.2. There is also a *universal* Fano scheme. The product:

$$\mathbf{F}(r,n,d) \subseteq \mathbf{G} \times \mathbf{P}^{N(d,n)}$$

carries the pull back of the tautological bundle $\pi_1^*\mathcal{S}$ and the global sections of the box product:

$$\mathrm{Sym}^d(\mathcal{S}^\vee) \boxtimes \mathcal{O}_{\mathbf{P}^{N(n,d)}}(1)$$

is the tensor product:

$$\mathrm{Sym}^d(k^{\oplus n+2})^\vee \otimes \mathrm{Sym}^d(k^{\oplus n+2}).$$

which has a natural section. The zero locus of this section cuts out the Fano scheme fiber by fiber over $\mathbf{P}^{N(n,d)}$.

Proposition 1.3. *The universal Fano scheme is a smooth, irreducible, projective variety of dimension:*

$$(r+1)\cdot(n+1-r) + \binom{n+1+d}{d} - \binom{r+d}{d} - 1.$$

Proof. This is a CLASSIC. Consider the fiber over a point $[\Lambda] \subseteq \mathbf{G}$. This consists of the hypersurfaces that contain the r-plane $\Lambda \subseteq \mathbf{P}$. This is the projective space

$$\mathbf{P}(\mathrm{H}^0(\mathbf{P}, I_\Lambda(d))).$$

Together these form a projective bundle over $\mathbf{G}$. This is then a dimension count (which I hopefully got right). □

Corollary 1.4. *The expected dimension of the Fano scheme $\mathbf{F}(X,r)$ is*

$$\dim(\mathbf{F}(X,r)) = (r+1)\cdot(n+1-r) - \binom{r+d}{d}.$$

If the universal Fano scheme dominates $\mathbf{P}^{N(r,d)}$ then for a general hypersurface this is true.

Before continuing, we recall the tangent space to the Fano scheme:

Theorem 1.5. *Let $\Lambda \subseteq X$ be an r-plane contained in a variety $X \subseteq \mathbf{P}^{n+1}$ which is smooth along L. Then*

$$T_{[\Lambda]}\mathrm{F}(X,r) = \mathrm{H}^0(\Lambda, \mathcal{N}_{\Lambda/X}).$$

Furthermore, if $\mathrm{H}^1(\Lambda, \mathcal{N}_{\Lambda/X}) = 0$ then $\mathrm{F}(X,r)$ is smooth at $[\Lambda]$.

Proof. Let us give a short sketch of the main idea. Let $\mathbf{D} = \mathrm{Spec}(k[\varepsilon])$ be the dual numbers and let $X_{\mathbf{D}} = X \times_k \mathbf{D}$ be the trivial family. We are looking to find flat families $\Lambda' \subseteq X_{\mathbf{D}}$ over $\mathbf{D}$ that have special fiber Λ. It is reasonable to think of (ε) and I_Λ as ideal sheaves on $X_{\mathbf{D}}$ by pulling them back as ideals. The ideal sum:

$$(\varepsilon) + I_\Lambda$$

is the ideal sheaf associated to the special fiber. For any flat family of interest Λ', it is not hard to check that

$$(\epsilon + I_\Lambda)^2 \subseteq I_{\Lambda'}.$$

The quotient

$$((\varepsilon) + I_\Lambda)/(\epsilon + I_\Lambda)^2$$

is the pushforward of a vector bundle on the special fiber Λ. That vector bundle is:

$$\varepsilon \cdot \mathcal{O}_\Lambda \oplus I_\Lambda/I_\Lambda^2$$

which has rank equal to $\mathrm{codim}(\Lambda \subseteq X) + 1$.

Choosing an ideal sheaf of Λ' in here amounts to choosing a subbundle of rank equal to $\mathrm{codim}(\Lambda \subseteq X)$. Moreover, when we quotient by ε this must project onto I_Λ / I_Λ^2. In otherwords, every such subbundle will be isomorphic to I_Λ / I_Λ^2 and choosing the subbundle amounts to choosing a homomorphism in

$$\mathrm{Hom}_{\mathcal{O}_\Lambda}(I_\Lambda / I_\Lambda^2, \mathcal{O}_\Lambda).$$

The smoothness hypothesis says this is isomorphic to:

$$\mathrm{Hom}_{\mathcal{O}_\Lambda}(\mathcal{O}_\Lambda, \mathcal{N}_{\Lambda/X}) \simeq \mathrm{H}^0(\Lambda, \mathcal{N}_{\Lambda/X}),$$

which gives the desired result.

We won't discuss the singularities, but you can check [Huy23] for the idea. $\qquad\square$

Now for a hypersurface $\Lambda \subseteq X \subseteq \mathbf{P}^{n+1}$ smooth along Λ, we have an exact sequence:

$$0 \to \mathcal{N}_{\Lambda/X} \to \mathcal{N}_{\Lambda/\mathbf{P}} \to \mathcal{N}_{X/\mathbf{P}}|_\Lambda \to 0.$$

This sequence becomes:

$$0 \to \mathcal{N}_{\Lambda/X} \to \mathcal{O}_\Lambda(1)^{\oplus n+1-r} \to \mathcal{O}_\Lambda(d) \to 0.$$

WLOG we can assume Λ is cut out by $x_{r+1}, \ldots, x_{n+1}$, in which case the final map is given by $\partial_{x_i} F$ $(r+1 \le i \le n+1)$. Numerically this implies:

$$\det(\mathcal{N}_{\Lambda/X}) = \mathcal{O}_\Lambda(n+1-r-d).$$

Corollary 1.6. *Let $L \subseteq X$ be a line in a cubic hypresurface $X \subseteq \mathbf{P}^{n+1}$ that is smooth along L. Then $\mathcal{N}_{L/X} = \mathcal{O}_L(a_1) \oplus \cdots \oplus \mathcal{O}_L(a_{n-1})$, $a_1 \ge \cdots \ge a_{n-1}$ with*

$$(a_1, \cdots, a_{n-1}) = \begin{cases}(1, \ldots, 1, 0, 0) & or \\ (1, \ldots 1, 1, -1).\end{cases}$$

Corollary 1.7. *The Fano scheme of lines $\mathrm{F}(X)$ of a smooth cubic hypersurface $X \subseteq \mathbf{P}^{n+1}$ is smooth and of dimenion $2n-4$ if non-empty. Moreover, the universal Fano scheme:*

$$\mathbf{F}(1, n, 3) \to \mathbf{P}^{N(n,3)}$$

is smooth over $U(n, 3)$.

References

[Huy23] Daniel Huybrechts. *The geometry of cubic hypersurfaces*, volume 206 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2023.